Balanced orbit defects in half-turn-symmetric no-three-in-line configurations, with new $2n$-point configurations for $n = 36$, 37, 39

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Abstract

A $2n$-point no-three-in-line configuration on the $n \times n$ grid with half-turn symmetry can be written as the union of the orbits of a larger subgroup $H \leq D_4$ (the quarter turns, or the two diagonal reflections) that it contains, together with a set of half-turn pairs that are not H-orbits — its *orbit defect* relative to H. We prove a *balance lemma*: the defect pairs, read as arcs on the row classes $\{i, n-1-i\}$, form a balanced directed multigraph, hence a union of directed cycles. With one further observation (two half-turn pairs on the same long diagonal are collinear) this classifies the defects with at most three pairs: one pair must be a diagonal loop — Flammenkamp's pseudo-class rct4; two pairs are two loops on different diagonals or a directed 2-cycle, which is an orbit of the diagonal reflections; three pairs form a directed 3-cycle or a loop plus a 2-cycle. We enumerate the corresponding families exhaustively for small n with an exact branch-and-bound program (validated against OEIS A000769 and the class counts of Flammenkamp's database): the 3-cycle family for odd $n \leq 25$ and $n = 33, 37, 39, 41$, each swept to completion in both bases (and, in part, 45); the two-loop family for even $n \leq 28$ and $n = 36$ ($n = 30, 32, 34$ not swept); the mixed families for $n \leq 27$ (all empty). Among the configurations found are: for $n = 36$, the first located $2n$-point configurations for that n whose symmetry group is exactly the half-turn (exactly three inequivalent ones, in the two-loop family); for $n = 37$ (two) and $n = 39$ (four), the first located configurations of that symmetry group outside the rct4 pseudo-class (Flammenkamp's rct4 configurations of 1997/98 already have exactly half-turn symmetry); and a further one for $n = 33$. "First located" refers to the public files of Flammenkamp's database as of 2026-08-11 (confirmed by its maintainer, who has added the configurations to the database). Configurations, programs and sweep journals are public.

1 Introduction

The no-three-in-line problem (Dudeney, 1917) asks for the largest set of points of the $n \times n$ grid $\{0, \ldots, n-1\}^2$ containing no three collinear points. Since a line parallel to an axis contains at most two points, at most $2n$ points can be placed, and $2n$-point configurations are the objects of interest. Their existence for every n is a well-known open problem; a probabilistic heuristic (Guy–Kelly, corrected form) suggests that for large n fewer than $2n$ points can be placed. Computationally, $2n$-point configurations were known for all $n \leq 46$ and for $n = 48, 50, 52$ for decades [1, 2]; in 2025–2026 constraint programming [4], GPU branch-and-bound [5] and SAT solving extended this to all $n \leq 70$ and to $n = 72, 74, 76$ (see the database [3], which also records, for each n, the number of configurations by their exact symmetry class under the dihedral group D_4 of the square, and marks the entries for which no configuration is known).

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This note concerns those entries rather than the record sizes. Flammenkamp's table lists, for each n, the number of inequivalent $2n$-point configurations whose stabilizer in D_4 is exactly a given subgroup: trivial (*iden*), the half-turn (*rot2*), one diagonal reflection (*dia1*), one axis reflection (*ort1*), the quarter turns (*rot4*), both diagonal reflections (*dia2*), both axis reflections (*ort2*) and the full group (*full*). Inside the half-turn class he keeps, for odd n, the pseudo-class *rct4* ("rot4 symmetry except on the long diagonals": quarter-turn orbits off the diagonals plus one half-turn pair on one diagonal), whose configurations have stabilizer exactly the half-turn but are listed in a separate column `c`, the column `:` containing the remaining rot2 configurations [3]. The rct4 configurations were found by Flammenkamp around the turn of 1996/97 and published in [2]; the odd-n construction of [4] is of the same type. As of 2026-08-11 the column `:` was complete for $n \le 31$; for $n = 36$ both columns were empty, and for $n = 37, 39$ the column `:` was empty while column `c` listed 21 and 33 configurations.

Results. (i) A balance lemma (Section 2) describing which orbit defects a half-turn-symmetric configuration may have relative to a larger subgroup $H \le D_4$: the defect pairs must form a balanced directed multigraph on the row classes. Together with the collinearity of two pairs on one diagonal this classifies the defects with at most three pairs (Proposition 3): a single pair is a diagonal loop (this is exactly the rct4 structure); two pairs are two loops on different diagonals or a directed 2-cycle, and a directed 2-cycle is itself an orbit of the diagonal reflections; three pairs form a directed 3-cycle or a loop plus a 2-cycle. (ii) Exhaustive enumerations of the resulting families for small n (Section 4), with all sub-classes listed and their journals kept: the 3-cycle family, the two-loop family and the mixed families. (iii) New $2n$-point configurations with stabilizer exactly the half-turn: three inequivalent ones for $n = 36$ (two-loop family; both database entries were empty), two for $n = 37$ and four for $n = 39$ (3-cycle family; the first located configurations of that stabilizer that are not of rct4 type), and one more for $n = 33$; the configurations are listed in the Appendix and were verified by independent programs. (iv) A description of the exact solver and its validation, measured growth of the cost of finding one $2n$-point configuration in the classes rot4, rct4 and rot2, and a list of negative results with their exact scope.

Related work. The best known lower bound, $3n/2 - o(n)$ points, is the modular-hyperbola construction of Hall, Jackson, Sudbery and Wild [7]; it has not been improved since 1975 (see [8, 9] and Problem 72 of [11]). For $k \ge 3$ points per line the asymptotic answer kn was obtained recently by probabilistic methods [8, 9], which explicitly do not apply to $k = 2$. The probabilistic heuristic of Guy and Kelly [6], in its corrected form [10, 4], predicts that $2n$-point configurations become rare for large n; Prellberg notes that the same heuristic wrongly predicts fewer than one symmetric configuration [4], and Flammenkamp explains the abundance of rotationally symmetric configurations by the fact that a pair of points blocks only $O(\log n)$ cells in the rotational classes against $\Theta(n)$ in the axis-reflection classes [3]. Machine-learning approaches (PatternBoost, reinforcement learning) have been compared with integer programming on this problem and do not scale beyond $n \approx 14$ [12]. The abstract content of the balance lemma — a zero-divergence condition and its cycle decomposition — is standard (see e.g. [13] for balanced two-coloured degree sequences and alternating cycles); its application to symmetric no-three-in-line configurations, and the resulting search families, we have not found in the literature. Flammenkamp uses "defect" for the shortfall from $2n$ points and calls a no-three-in-line set with exactly $2n$ points a *solution*; to avoid a clash we say *orbit defect*, and our "$2n$-point configuration" is his "solution".

2 Symmetry classes and the balance lemma

Let $m = n-1$ and let D_4 act on the grid; write $\rho(u,v) = (v, m-u)$ for the quarter turn, so $\rho^2(u,v) = (m-u, m-v)$, $\sigma(u,v) = (v,u)$ for the reflection in the main diagonal and $\sigma'(u,v) = (m-v, m-u) = \rho^2\sigma(u,v)$ for the reflection in the anti-diagonal. A $2n$-point no-three-in-line configuration has exactly two points in every row and every column.

Fix a subgroup $H \le D_4$ containing ρ^2 and a configuration C with $\rho^2 C = C$. Let B be the union of all H-orbits contained in C (the *H-base* of C; it is H-invariant, and it is determined by C and H). Since C is ρ^2-invariant and $\rho^2 \in H$, the complement $D = C \setminus B$ is a union of half-turn pairs $\{p, \rho^2 p\}$, none of which is an H-orbit; D is the *orbit defect* of C relative to H. For $a \in \{0, \dots, m\}$ let $r_X(a)$, $c_X(a)$ denote the numbers of points of a set X in row a and in column a.

Lemma 1 (balance)**.** *Suppose that H contains a motion g with $g(\{\text{column } a\}) = \{\text{row } a\}$ for every a (the quarter turn ρ and the reflection σ have this property; note that σ' alone maps column a onto row $m-a$, but $\sigma' \in H$ implies $\sigma = \rho^2\sigma' \in H$). Then every H-orbit O satisfies $r_O(a) = c_O(a)$ for all a, and consequently the orbit defect D of a $2n$-point configuration satisfies $r_D(a) = c_D(a)$ for all a.*

Proof. g maps $O \cap \{\text{column } a\}$ bijectively onto $O \cap \{\text{row } a\}$ because $gO = O$; hence $r_O(a) = c_O(a)$. Summing over the orbits in B and using $r_C(a) = c_C(a) = 2$ gives $r_D(a) = c_D(a)$. ☐

Interpret the pairs of D as arcs of a directed multigraph on the *row classes* $\bar{x} = \{x, m-x\}$ (for odd n the central class $\{m/2\}$ is a singleton): the pair $\{(x,y), (m-x, m-y)\}$ occupies rows $x, m-x$ and columns $y, m-y$, i.e. it is an arc from $\bar{x}$ to $\bar{y}$ (a loop if $\bar{x} = \bar{y}$; for $\bar{x} \ne \bar{y}$ two different pairs can join the same two classes). A pair with $x = m/2$ has both points in the central row and is an arc out of the central class; a pair with $y = m/2$ is an arc into it; the central *cell* $(m/2, m/2)$ cannot belong to a ρ^2-invariant configuration with two points per row, but the central class may well be used. Then $r_D(\bar{x}) = 2 \cdot \operatorname{outdeg}(\bar{x})$ and $c_D(\bar{x}) = 2 \cdot \operatorname{indeg}(\bar{x})$, and Lemma 1 says:

Corollary 2. *The orbit defect D is a balanced directed multigraph on the row classes (in-degree = out-degree at every class), hence its arc set decomposes into directed cycles (loops and 2-cycles allowed). In particular a single defect pair must be a loop, i.e. must lie on a long diagonal. For $H = \langle\rho\rangle$ every H-orbit in C has four points (the centre cell cannot occur), so $2n = 4a + 2|D|$ and $|D| \equiv n \pmod 2$; for $H = \langle\sigma, \sigma'\rangle$ the diagonal 2-orbits $\{(i,i), (m-i, m-i)\}$, $\{(i, m-i), (m-i, i)\}$ are H-orbits, and if b of them occur then $b + |D| \equiv n \pmod 2$.*

For $H = \langle\rho\rangle$ (rot4) an H-orbit off the diagonals is $\{(u,v), (v, m-u), (m-u, m-v), (m-v, u)\}$; for $H = \langle\sigma, \sigma'\rangle$ (dia2) it is $\{(u,v), (v,u), (m-u, m-v), (m-v, m-u)\}$. We call these the C4 base and the V2 base. The following proposition lists all orbit defects with at most three pairs; the families it names are then the natural next steps beyond rct4.

Proposition 3 (orbit defects with at most three pairs)**.** *Let C be a $2n$-point configuration with $\rho^2 C = C$ and let D be its orbit defect relative to $H = \langle\rho\rangle$.*

(a) $|D| = 1$: D is one loop, i.e. one half-turn pair on a long diagonal (n odd) — the rct4 structure.

*(b) $|D| = 2$ (n even): either two loops at different classes on different diagonals, $\{(x,x), (m-x, m-x)\}$ and $\{(y, m-y), (m-y, y)\}$ with $\bar{y} \ne \bar{x}$ (**two-loop family**); or a directed 2-cycle $\bar{x} \to \bar{y} \to \bar{x}$, which is a dia2-orbit $\{(u,v), (v,u), (m-u, m-v), (m-v, m-u)\}$ with $\bar{u} = \bar{x}$, $\bar{v} = \bar{y}$ that is not a C4-orbit (**mixed family** C4 base + one V2 orbit).*

(c) $|D| = 3$ (*n odd*): *either a directed* 3*-cycle on three distinct classes,* $(x, y), (y', z), (z', x')$ *with* $\bar{y}' = \bar{y}$, $\bar{z}' = \bar{z}$, $\bar{x}' = \bar{x}$, *one of the classes possibly the central one* (3**-cycle family**)*; or one loop plus a directed* 2*-cycle, i.e. a diagonal pair plus a dia2-orbit that is not a C4-orbit* (**mixed family with a loop**).

The same list, with the roles of C4- and dia2-orbits exchanged, holds for $H = \langle \sigma, \sigma' \rangle$*; there a directed* 2*-cycle is a C4-orbit that is not a dia2-orbit* (**mixed family** *V2 base + one C4 orbit*)*, and a loop is a diagonal* 2*-orbit and hence never a defect.*

Proof. By Corollary 2, D is a union of directed cycles on the classes. Two loops on the same diagonal give four collinear points, and two loops at the same class $\bar{x}$ on the two diagonals form the C4-orbit of (x, x), which belongs to the base by definition; this leaves (a) and the first case of (b), and excludes three loops. A directed 2-cycle $\bar{x} \to \bar{y} \to \bar{x}$ consists of two pairs $\{(u, v), (m-u, m-v)\}$ and $\{(v', u'), (m-v', m-u')\}$ with $\bar{u} = \bar{u}' = \bar{x}$, $\bar{v} = \bar{v}' = \bar{y}$; the second pair is the image of the first under σ (if $(v', u') \in \{(v, u), (m-v, m-u)\}$) or under ρ (if $(v', u') \in \{(v, m-u), (m-v, u)\}$); in the second case the union is the C4-orbit of (u, v) and belongs to the base, in the first case it is the dia2-orbit of (u, v). For $|D| = 3$ the balanced multigraphs are a 3-cycle, a loop plus a 2-cycle, or three loops. For the V2 base the argument is the same with ρ and σ exchanged. □

A configuration of one of these families is not H-symmetric (the defect breaks H); its stabilizer in D_4 is some subgroup containing ρ^2 and not containing H, and it must be computed: for every configuration reported below the stabilizer was determined by testing all eight motions, and it is exactly $\{1, \rho^2\}$ in every case, so the configurations fall into the rot2 column of the database. (For n even, a two-loop configuration is "rot4 symmetric except on the long diagonals"; the database keeps rct4 as a pseudo-class for odd n only, so these configurations belong to the column :.)

3 The solver

All searches were done with an exact branch-and-bound program (`there.c`, about 600 lines of C, no external libraries; the family sweeps used the variant `there_twisted_experiment.c` with the two bases and the defect built in). Cells are $t = un + v$; a state keeps two bit-matrices, *alive* by rows and by columns. Every pair of chosen points "shades" all grid cells of the line through them (walked with the primitive direction obtained from a gcd table); shaded cells die, as do all cells of a row or column that already holds two points. A symmetry class is imposed by choosing whole orbits: a cell is a candidate only if its whole orbit is alive, and the "orbit-alive" bit-masks are derived from the alive matrices by transposition and bit reversal (one word operation per group element). Branching uses the most constrained reading class (row or column with the fewest orbit-alive cells) and dies as soon as some class cannot be completed. A failed candidate is killed for its siblings, so that a search with a fixed symmetry class is complete: "no configuration" is a computer-assisted proof of absence for that class (trusted program, no independent certificate). With restarts and randomized tie-breaking the same program is used to find one configuration quickly. A `PAIRS` option fixes a set of half-turn pairs (the orbit defect), kills their images under the base group, and searches the remaining orbits of the base; the same option with a diagonal pair reproduces rct4 exactly. Invalid `PAIRS/FIX` input is rejected.

Validation. For $n \leq 10$ the program enumerates all $2n$-point configurations without symmetry: $1, 2, 11, 32, 50, 132, 380, 368, 1135$ labelled configurations for $n = 2, \ldots, 10$, which reduce to $1, 1, 4, 5, 11, 22, 57, 51, 156$ classes under D_4, in agreement with OEIS A000769. For symmetric classes we compared the number N_H of labelled configurations invariant under a fixed subgroup H with the class counts c_K of [3]. For a normal subgroup H (rot2, rot4, dia2, ort2, full) all $|D_4|/|K|$ labelled images of a configuration of class $K \supseteq H$ are H-invariant, so $N_H = \sum_{K \supseteq H} c_K \, |D_4|/|K|$:

rot2 for $n = 8$: $36 = 4 \cdot 7 + 2 \cdot 4$; $n = 10$: $67 = 4 \cdot 13 + 2 \cdot 6 + 2 \cdot 1 + 1$; dia2 for $n = 11, \ldots, 25$: $0, 2, 2, 0, 0, 0, 2, 4 = 2 \cdot (0, 1, 1, 0, 0, 0, 1, 2)$. For $H = \langle\sigma\rangle$ (dia1) only two of the four images of a dia1 configuration have stabilizer $\langle\sigma\rangle$ (the other two have $\langle\sigma'\rangle$), so $N_H = 2c_{\text{dia1}} + 2c_{\text{dia2}} + c_{\text{full}}$: $n = 13$: $20 = 2 \cdot 9 + 2 \cdot 1$; $n = 15$: $28 = 2 \cdot 13 + 2 \cdot 1$. For rct4 the program's mode places the diagonal pair on the main diagonal and therefore sees two of the four labelled images of each rct4 configuration: $n = 9, 17, 19, 21$: $2, 2, 4, 2 = 2 \cdot (1, 1, 2, 1)$ — all in agreement. Two independent implementations (the two agents' programs) agree on every enumeration they both performed. In addition, the family modes were checked against a from-scratch enumerator written independently of the solver (`slack/t221/verify_family.py`: it builds the sub-class as "a union of base-group orbits plus the prescribed half-turn pairs" and tests all triples in exact arithmetic): for $n = 13$ all 20 sub-classes tried in the C4 base and all 13 in the V2 base agree, including the 25 that contain no configuration — the direction in which an incomplete search would fail. We note for the record that the solver's plain *rot2* mode (`sym=2`) is a search mode, not an enumerator: at $n = 8$ it returns 8 of the 36 labelled half-turn-invariant configurations, and it was not used for any statement in this paper. Every configuration reported here was finally re-checked by an independent script that tests all $\binom{2n}{3}$ triples in exact integer arithmetic and computes the stabilizer explicitly.

4 Results

4.1 Exhaustive enumerations of the defect families

The defect sets were enumerated up to D_4 (one representative per orbit of D_4 acting on defects, computed as canonical forms of the six or four cells); for each representative the sub-class (defect + base) was searched exhaustively. Table 1 gives, for odd n, the number of inequivalent 3-cycle defects — $4\binom{h}{3} + \binom{h}{2}$ with $h = (n-1)/2$, the second term being the cycles through the central class — and the number of pairwise inequivalent configurations found for the two bases; Table 2 gives the two-loop family for even n; the mixed families are reported in the text. Rows marked "complete" were swept entirely, every sub-class terminating with "exhausted" (or "dead", when the fixed defect already violates the law: e.g. 24 of the 2240 non-central sub-classes at $n = 33$); counts marked with $\geq$ come from sweeps that were still running when this draft was written, with the number of sub-classes already swept stated. *Correction to an earlier draft:* the generator of the 3-cycle lists used until v0.6 omitted the cycles through the central class (an external audit pointed this out); the omitted sub-classes were swept afterwards and produced two additional configurations at $n = 13$ (V2 base) and $n = 21$ (C4 base) — both already in the database, whose rot2 column is complete for $n \leq 31$ — none at $n = 15, \ldots, 25$, 33 and 37, and two new ones at $n = 39$; the corrected counts are given below, and all journals are in the repository (`logs/`).

Mixed families. The mixed families of Proposition 3 — C4 base plus one dia2-orbit that is not a C4-orbit (plus one diagonal loop for odd n), and V2 base plus one C4-orbit that is not a dia2-orbit — were swept exhaustively for all $9 \leq n \leq 27$ (both parities): no configuration exists in them. Across the range this is 8508 sub-classes mod D_4, of which 320 are *dead* — the defect cells already contain three collinear points — and hence empty a fortiori; the remaining 8188 were each exhausted by search. For example, at $n = 23$ there are 1210 sub-classes (50 dead) and 55 on the V2 side; at $n = 27$, 2028 (72 dead) and 78. *Correction to an earlier draft:* the counts quoted here previously mixed two conventions — the total 1210 at $n = 23$ but the live count $1956 = 2028 - 72$ at $n = 27$; an external review noticed the inconsistency, and the difference of exactly 72 was traced to the dead sub-classes. All counts above are totals, dead included and stated.

The emptiness is quantifiable. On the same range of n the 3-cycle family carries ten configurations over 6132 swept sub-classes, one per 613; at that rate the 8188 live mixed sub-

n	defects mod D_4 (central)	C4 base	V2 base	status of the sweep
13	95 (15)	5	3	complete
15	161 (21)	0	0	complete
17	252 (28)	0	1	complete
19	372 (36)	0	0	complete
21	525 (45)	1	0	complete
23	715 (55)	0	0	complete
25	946 (66)	0	0	complete
33	2360 (120)	1	0	complete (both bases)
37	3417 (153)	2	0	complete (both bases)
39	4047 (171)	4	0	complete (both bases): C4 base 4 configurations (2 through the central class), V2 base none
41	4750 (190)	0	0	complete (both bases): every one of the 4750 sub-classes exhausted in each base, no configuration
45	6391 (231)	—	≥ 0	V2 base: 400 sub-classes swept (none); C4 base not started

Table 1: The 3-cycle family: C4- or V2-orbits plus a directed 3-cycle of half-turn pairs. The parity $|D| \equiv n \pmod 2$ forces odd n for the C4 base only; for the V2 base a 3-cycle at *even* n (with b odd) is permitted by the parity relation and has not been swept — an open family, named here rather than left implicit. Numbers below: numbers of pairwise inequivalent configurations. A sub-class is one defect (one directed 3-cycle up to D_4) together with one base; "complete" means every sub-class was searched exhaustively.

classes would be expected to carry about 13, and none was found ($e^{-13.4} \approx 1.6 \cdot 10^{-6}$ under a naive Poisson reading). The comparison is heuristic — sub-classes of different families are not exchangeable — but it is enough to say that the emptiness looks structural rather than accidental. We still have no structural explanation, and now know one is worth seeking.

4.2 New configurations with exactly half-turn symmetry

Table 3 lists the new configurations; their coordinates and their encodings in the 90-character format of [3] are given in the Appendix. "First located" refers to the public files of the database on 2026-08-11 and to a search of the literature; the maintainer has since confirmed that the : entries for $n = 36, 37, 39$ were empty on that date and has added the configurations of Table 3 to the database. For $n = 36$ both database entries were empty; for $n = 37$ and 39 the column `c` (rct4) was populated, so the configurations below are new representatives of exact half-turn symmetry outside the rct4 structure, not the first configurations with that stabilizer; the $n = 33$ configuration is new but the entry was not empty. All ten configurations were checked to be pairwise inequivalent (canonical forms under D_4). The second and third $n = 39$ configurations have their defects through the central class — they were found only after the correction of the generator described above; the fourth was found when the exhaustive C4-base sweep of $n = 39$ completed (18 August 2026) and is not yet in the database.

4.3 Cost of finding one configuration, and growth

For orientation we report measured costs of finding one $2n$-point configuration on the $n \times n$ grid with the solver of Section 3, on two consumer machines (Apple M3 laptop, 8 cores; a 12-core x86 server; 8–12 seeds in parallel; wall time to the first configuration): rot4 (even n): $n = 36$: 10–60 s, $n = 40$: 70 s; rct4 (odd n): $n = 41$: 806 s, $n = 43$: about 6.7 core-hours, $n = 45$: none within 3.3 core-hours; rot2: $n = 24$: 10 s, $n = 27$: 137 s. These are measured ratios for named instances,

n	sub-classes $\{x,y\}$ mod D_4	inequivalent configurations
14–22	21, 28, 36, 45, 55	0 (complete)
24	66	2 (complete)
26	78	1 (complete)
28	91	1 (complete)
36	153	3 (complete: three sub-classes contain a configuration, two labelled solutions each, i.e. one configuration up to the diagonal reflections; the sub-class $\{1,4\}$ needed a rerun without time limit, which exhausted it in $4.84 \cdot 10^7$ nodes with no configuration; the rerun is recorded in `logs/sweeps/twoloop_n36_exh.txt`)

Table 2: The two-loop family (even n): C4-orbits plus one half-turn loop on each long diagonal. The gap $n = 30, 32, 34$ is a gap in the sweeping budget, not an emptiness claim. A sub-class is an unordered pair of row classes $\{x, y\}$, $0 \le x < y < n/2$ (x = class of the main-diagonal loop, y = class of the anti-diagonal loop, or vice versa: the quarter turn interchanges the two roles), so there are $n(n-2)/8$ sub-classes.

n	family and defect	remark
36	two loops: $(6,6),(29,29)$ and $(5,30),(30,5)$	first located exact half-turn (both cells empty)
36	two loops: $(1,1),(34,34)$ and $(9,26),(26,9)$	second, inequivalent
36	two loops: $(1,1),(34,34)$ and $(7,28),(28,7)$	third, inequivalent
37	3-cycle $(1,3),(3,31),(31,1)$, C4 base	first located outside rct4 (column :)
37	3-cycle $(2,4),(4,20),(20,34)$, C4 base	second, inequivalent
39	3-cycle $(4,8),(8,20),(20,34)$, C4 base	first located outside rct4 (column :)
39	3-cycle $(1,5),(5,19),(19,1)$ through the central class, C4 base	second, inequivalent
39	3-cycle $(4,7),(7,19),(19,4)$ through the central class, C4 base	third, inequivalent
39	3-cycle $(1,3),(35,14),(24,1)$, C4 base	fourth, inequivalent (found by the exhaustive C4-
33	3-cycle $(2,3),(3,19),(19,2)$, C4 base	new (entry not empty)

Table 3: New $2n$-point configurations with stabilizer exactly $\{1, \rho^2\}$.

not a complexity law: the cost grows by roughly a factor 8–10 per +4 in n for rot4/rct4 and by a factor of about 6 per +2 for rot2, consistent with the exhaustive tree sizes (rot4: $\times 20$ per +4; rot2 and dia1: $\times$9–10 per +2) divided by the slowly growing numbers of configurations, and consistent with the growth reported for CP-SAT in [4]. Extrapolated — with the obvious uncertainty of an extrapolation over more than 25 steps — to the open sizes $71, 73, 75, 77, \ldots$ this suggests 10^5–10^7 core-hours for our program; the record computations of 2026 (SAT and GPU search on clusters) are of that order or use better tools. Sub-class sweeps of the defect families are cheap because a fixed defect shades a large part of the grid: a sub-class of the 3-cycle family at n = 37–41 is exhausted in seconds to a few minutes.

4.4 Negative results, with their scope

- *Consequence of the lemma.* A single defect pair off the diagonals is impossible (Corollary 2); the “rot4 except one pair anywhere” family is empty, as our enumerations for $n \le 27$ had shown before the lemma was found.
- *Computer-assisted, exhaustive.* The mixed families are empty for $9 \le n \le 27$ (above); the two-loop family is empty for $14 \le n \le 22$; the 3-cycle family is empty for $n = 15, 19, 23, 25$.
- *Bounded search, no conclusion about existence.* Shifting a known rot4 configuration for $n = 44$ into the 46×46 grid (which preserves all its collinearities) and freeing k of its 22 orbits, no completion exists for $k \le 15$ (exhaustive for those k); local search on exactly $2n$ points (min-conflicts with tabu, simulated annealing) failed from $n \approx$ 12–28 on with

our implementations and budgets, although it produces large partial configurations easily (e.g. 134 points on the 71×71 grid); belief-propagation decimation without backtracking failed from $n = 8$ on. These are statements about the specified methods, not about the configurations.

- *Empirical observation.* For the rot4/rct4 configurations of the database with $41 \le n \le 76$ (142 configurations) the largest subset lying on one line, axis-parallel parabola or shifted hyperbola over $\mathbb{F}_p$, maximised over primes $11 \le p \le 61$, is on average 16% of the points (never above 22%), *less* than for random sets with two points per row (19% on average): relative to these curve families and primes, the finite extremal configurations do not reduce modulo a prime to a low-degree curve (cf. Green's remark on Problem 72 of [11]), while the best asymptotic constructions do. On the algebraic side, the modular-hyperbola construction of [7] is optimal within its own $2p \times 2p$ box, unions of two hyperbolae gain only a bounded amount as far as computed, and for $H(1) \cup H(-1)$ a lawful subset has at most $(11/3 + o(1))(p-1)$ points (an explicit line cover); see the companion note [14].

5 Reproducibility

The programs, the sub-class lists, the sweep journals and the certified configurations are in the public repository `https://github.com/iwasborninbali/saturation` (immutable tag `defects-v1.1`; checksums in `docs/RELEASES.md`):
`there.c`; `kit71/there_twisted_experiment.c` and its even-n variant `bench/there_tw_even.c` (the two bases and the `PAIRS` defect built in; validated against full enumerations of the rot4 and dia2 classes); the generators `bench/defects3.py` (all 3-cycle defects mod D_4, central class included) and `bench/mixed3.py`; the sweep drivers `bench/family_sweep2.sh`, `central_sweep.sh`, `mixed_sweep.sh`, `twoloop_exh.sh` (same directory); the finished journals under `logs/sweeps/` (one line per sub-class with the solver's termination status; `BOOK` lines with every labelled solution); `web/stab.py` for the exact class and the database encoding; `docs/SUBMISSION.md` for the configurations; `make check` for a one-command test. Typical commands: `ALL=1 PAIRS="1,3;3,31;31,1" ./there 37 2` enumerates the sub-class that contains the first $n = 37$ configuration (one configuration, $1.4 \cdot 10^7$ nodes); `PAIRS="6,6;5,30" ./there 36 2` finds the first $n = 36$ configuration in about 100 s and `PAIRS="1,1;9,26" ./there 36 2` the second in about 30 s (wall time on the laptop); the sub-class $\{5, 6\}$ is exhausted in about 7 min and contains exactly two labelled configurations, i.e. one up to the diagonal reflections, which preserve both loops. Journals of the early sweeps run on the first agent's machine (the first halves of the lists for $n = 39, 41, 45$) are being added; the $n = 33$ row was rerun in full with the corrected generator on a cloud machine (`logs/sweeps/family3full_*_n33.txt`: 2360 sub-classes per base, one configuration), and the $n = 37$ row was completed by sweeping the 153 central sub-classes for both bases. All exhaustive statements are computer-assisted results of a trusted program (validated as in Section 3); no independent certificates of non-existence are produced.

6 AI/tool-use disclosure and authorship statement

Following the arXiv policy on generative AI, the AI systems involved are not listed as authors; their role is disclosed here. This work was carried out by autonomous AI agents (Claude Fable 5, Anthropic), running as independent solvers on separate machines and communicating through the repository, under the direction of the author of record; a third agent of the same kind worked mainly on the companion note [14]. The agents wrote the programs, designed and ran the searches, found and proved the balance lemma (independently of each other, in two formulations), verified each other's results, and drafted and revised this text, including the corrections after two external LLM-assisted audits commissioned by the author of record; the author of record posed the problem, provided the computing environment, made all decisions about scope and publication, and takes responsibility for the content; the extent of his independent verification of the proofs and of the computations is recorded in `docs/HUMAN_AUDIT.md` of the repository and will be stated precisely in any submitted version. We estimate that about nine tenths of the technical work was done by the AI systems; we state this plainly because it is true. All configurations reported here are verifiable by anyone with the scripts in the repository (every triple checked in exact integer arithmetic).

Acknowledgements

We thank Achim Flammenkamp for maintaining the no-three-in-line database and its class counts, without which the empty cells would not have been visible, and Thomas Prellberg for the detailed account of the CP-SAT computations.

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A The configurations

Rows and columns are numbered $0, \ldots, n-1$ from the top-left corner; the encoding is the class character followed by two column characters per row, with the 90-character alphabet of [3]:

`0-9A-Za-z#$%&@?!()[]<>{}=*+|-/~^_:;,.`

$n = 36$ **(configuration 1), 72 points, stabilizer** $\{1, \rho^2\}$**.** Encoding:
`:3NEHFSVZ3FPU6E2BNQ8I5GDS08JOTYVXDP9Y1QAM2416BGRZ7MJUHR9COXLT5AKW047KILCW`
Rows $u : v_1, v_2$ (row u has points at columns v_1, v_2):
0: 3,23 1: 14,17 2: 15,28 3: 31,35 4: 3,15 5: 25,30 6: 6,14 7: 2,11
8: 23,26 9: 8,18 10: 5,16 11: 13,28 12: 0,8 13: 19,24 14: 29,34 15: 31,33
16: 13,25 17: 9,34 18: 1,26 19: 10,22 20: 2,4 21: 1,6 22: 11,16 23: 27,35
24: 7,22 25: 19,30 26: 17,27 27: 9,12 28: 24,33 29: 21,29 30: 5,10 31: 20,32
32: 0,4 33: 7,20 34: 18,21 35: 12,32

$n = 36$ **(configuration 2), 72 points, stabilizer** $\{1, \rho^2\}$**.** Encoding:
`:JK1J9OCK6AORRVEH56QXLV25MWCIAS0301DS7MYZWZ7PHN3DUX4E29TUIL488BPTFNBQGYFG`
Rows $u : v_1, v_2$ (row u has points at columns v_1, v_2):
0: 19,20 1: 1,19 2: 9,24 3: 12,20 4: 6,10 5: 24,27 6: 27,31 7: 14,17
8: 5,6 9: 26,33 10: 21,31 11: 2,5 12: 22,32 13: 12,18 14: 10,28 15: 0,3
16: 0,1 17: 13,28 18: 7,22 19: 34,35 20: 32,35 21: 7,25 22: 17,23 23: 3,13
24: 30,33 25: 4,14 26: 2,9 27: 29,30 28: 18,21 29: 4,8 30: 8,11 31: 25,29
32: 15,23 33: 11,26 34: 16,34 35: 15,16

$n = 36$ **(configuration 3), 72 points, stabilizer** $\{1, \rho^2\}$**.** Encoding:
`:DK1GGNLRBK7RNPSU35AH6QIV26LZ3D04XYBQ9O12VZMW0ETX4H9TIPUW57AC8SFO8ECJJYFM`
Rows $u : v_1, v_2$ (row u has points at columns v_1, v_2):
0: 13,20 1: 1,16 2: 16,23 3: 21,27 4: 11,20 5: 7,27 6: 23,25 7: 28,30
8: 3,5 9: 10,17 10: 6,26 11: 18,31 12: 2,6 13: 21,35 14: 3,13 15: 0,4
16: 33,34 17: 11,26 18: 9,24 19: 1,2 20: 31,35 21: 22,32 22: 0,14 23: 29,33
24: 4,17 25: 9,29 26: 18,25 27: 30,32 28: 5,7 29: 10,12 30: 8,28 31: 15,24
32: 8,14 33: 12,19 34: 19,34 35: 15,22

$n = 37$ **(configuration 1), 74 points, stabilizer** $\{1, \rho^2\}$**.** Encoding:
`:GJ39EP7VENDZJRSX7P6ZCF28IQ4VWYKQFa06COUa0LAG245WAISYLO1UBT389H1NDM5TBMRXHK`
Rows $u : v_1, v_2$ (row u has points at columns v_1, v_2):
0: 16,19 1: 3,9 2: 14,25 3: 7,31 4: 14,23 5: 13,35 6: 19,27 7: 28,33
8: 7,25 9: 6,35 10: 12,15 11: 2,8 12: 18,26 13: 4,31 14: 32,34 15: 20,26
16: 15,36 17: 0,6 18: 12,24 19: 30,36 20: 0,21 21: 10,16 22: 2,4 23: 5,32
24: 10,18 25: 28,34 26: 21,24 27: 1,30 28: 11,29 29: 3,8 30: 9,17 31: 1,23
32: 13,22 33: 5,29 34: 11,22 35: 27,33 36: 17,20

$n = 37$ **(configuration 2), 74 points, stabilizer** $\{1, \rho^2\}$**.** Encoding:
`:FJ7E4PBGKO6U5VDZDH9RCF2X4QSTIZQa2X0SEM8a3Y0A1I78AW3YLO9RJN1N5V6UCGKPBWMTHL`
Rows $u : v_1, v_2$ (row u has points at columns v_1, v_2):
0: 15,19 1: 7,14 2: 4,25 3: 11,16 4: 20,24 5: 6,30 6: 5,31 7: 13,35

8: 13,17   9: 9,27   10: 12,15   11: 2,33   12: 4,26   13: 28,29   14: 18,35   15: 26,36
16: 2,33   17: 0,28   18: 14,22   19: 8,36   20: 3,34   21: 0,10   22: 1,18   23: 7,8
24: 10,32   25: 3,34   26: 21,24   27: 9,27   28: 19,23   29: 1,23   30: 5,31   31: 6,30
32: 12,16   33: 20,25   34: 11,32   35: 22,29   36: 17,21

$n = 39$ **(configuration 1), 78 points, stabilizer** $\{1, \rho^2\}$**.** Encoding:
`:CGAWLU7M8CAN1FPZ2K9TXbOPYc7BBL5W3c2E4JIKJYOa0Z6XHRRV04DE159TIa3DNbFSQUGV8H6SMQ`
Rows $u : v_1, v_2$ (row u has points at columns v_1, v_2):
0: 12,16   1: 10,32   2: 21,30   3: 7,22   4: 8,12   5: 10,23   6: 1,15   7: 25,35
8: 2,20   9: 9,29   10: 33,37   11: 24,25   12: 34,38   13: 7,11   14: 11,21   15: 5,32
16: 3,38   17: 2,14   18: 4,19   19: 18,20   20: 19,34   21: 24,36   22: 0,35   23: 6,33
24: 17,27   25: 27,31   26: 0,4   27: 13,14   28: 1,5   29: 9,29   30: 18,36   31: 3,13
32: 23,37   33: 15,28   34: 26,30   35: 16,31   36: 8,17   37: 6,28   38: 22,26

$n = 39$ **(configuration 2), 78 points, stabilizer** $\{1, \rho^2\}$**.** Encoding:
`:BI5HCWEQST8J2MSU7X4M47Dc3aKRNZEH69NbDc1bOP1FTWLO3FBI2ZOPVYGY5V8AGaJU9ACO6QLXKR`
Rows $u : v_1, v_2$ (row u has points at columns v_1, v_2):
0: 11,18   1: 5,17   2: 12,32   3: 14,26   4: 28,29   5: 8,19   6: 2,22   7: 28,30
8: 7,33   9: 4,22   10: 4,7   11: 13,38   12: 3,36   13: 20,27   14: 23,35   15: 14,17
16: 6,9   17: 23,37   18: 13,38   19: 1,37   20: 0,25   21: 1,15   22: 29,32   23: 21,24
24: 3,15   25: 11,18   26: 2,35   27: 0,25   28: 31,34   29: 16,34   30: 5,31   31: 8,10
32: 16,36   33: 19,30   34: 9,10   35: 12,24   36: 6,26   37: 21,33   38: 20,27

$n = 39$ **(configuration 3), 78 points, stabilizer** $\{1, \rho^2\}$**.** Encoding:
`:AEIUHR9P678GRYJP1XHZNc26CQ37NcAEIXTaMb4Y1G295KOSOFVZCQWaOF3L5bDJ4BMUVWDTBL8KOS`
Rows $u : v_1, v_2$ (row u has points at columns v_1, v_2):
0: 10,14   1: 18,30   2: 17,27   3: 9,25   4: 6,7   5: 8,16   6: 27,34   7: 19,25
8: 1,33   9: 17,35   10: 23,38   11: 2,6   12: 12,26   13: 3,7   14: 23,38   15: 10,14
16: 18,33   17: 29,36   18: 22,37   19: 4,34   20: 1,16   21: 2,9   22: 5,20   23: 24,28
24: 0,15   25: 31,35   26: 12,26   27: 32,36   28: 0,15   29: 3,21   30: 5,37   31: 13,19
32: 4,11   33: 22,30   34: 31,32   35: 13,29   36: 11,21   37: 8,20   38: 24,28

$n = 39$ **(configuration 4), 78 points, stabilizer** $\{1, \rho^2\}$**.** Encoding:
`:DH3C4SLOVa6FQX4MKMAD2TFI6bTcJbRX783c8REOBUOZUV5B1JO91WKN9aPSGIGY5CNW27EHAYQZLP`
Rows $u : v_1, v_2$ (row u has points at columns v_1, v_2):
0: 13,17   1: 3,12   2: 4,28   3: 21,24   4: 31,36   5: 6,15   6: 26,33   7: 4,22
8: 20,22   9: 10,13   10: 2,29   11: 15,18   12: 6,37   13: 29,38   14: 19,37   15: 27,33
16: 7,8   17: 3,38   18: 8,27   19: 14,24   20: 11,30   21: 0,35   22: 30,31   23: 5,11
24: 1,19   25: 0,9   26: 1,32   27: 20,23   28: 9,36   29: 25,28   30: 16,18   31: 16,34
32: 5,12   33: 23,32   34: 2,7   35: 14,17   36: 10,34   37: 26,35   38: 21,25

$n = 33$**, 66 points, stabilizer** $\{1, \rho^2\}$**.** Encoding:
`:MOAC3H9J6GJOBS9BO5PTOVPQHV5UEI2C4SKUEI2R1F671W37RWLN4L8DGQDNFTKM8A`
Rows $u : v_1, v_2$ (row u has points at columns v_1, v_2):
0: 22,24   1: 10,12   2: 3,17   3: 9,19   4: 6,16   5: 19,24   6: 11,28   7: 9,11
8: 0,5   9: 25,29   10: 0,31   11: 25,26   12: 17,31   13: 5,30   14: 14,18   15: 2,12
16: 4,28   17: 20,30   18: 14,18   19: 2,27   20: 1,15   21: 6,7   22: 1,32   23: 3,7
24: 27,32   25: 21,23   26: 4,21   27: 8,13   28: 16,26   29: 13,23   30: 15,29   31: 20,22
32: 8,10